LETTER TO THE EDITOR

A hierarchical model for ageing

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Abstract.   We present a one-dimensional model for diffusion on a hierarchical tree structure. It is shown numerically that this model exhibits ageing phenomena although no disorder is present. The origin of ageing in this model is therefore the hierarchical structure of phase space.

Strongly disordered systems have been a focal point of research in recent years. Among the most studied materials are magnetic systems with impurities and especially spin glasses [1]. Their dynamics can be characterized by glassy features which are also encountered in many different systems such as structural glasses [2], polymers in random potential [3], protein folding [4], dirty superconductors [5], charge density waves with impurities [6] and also areas like biological evolution [7].

Spin glasses are magnetic materials with structural disorder, e.g. alloys of a magnetic and a non-magnetic metal as for example AgMn and CuMn. An overview of different spin glass materials can be found in [1]. In the beginning, 20 years ago, the main interest was to find the equilibrium properties of spin glasses, since it was expected that a physical system reaches its equilibrium in finite time. In spin glasses, however, the dynamics is governed by non-equilibrium properties such as *ageing* which was found almost ten years later [8].

In this context ageing describes the striking effect that magnetic properties in spin glasses depend drastically on their history (or age) in the frozen phase. Although it is mainly referred to ageing in the context of spin glasses it can also be observed in other disordered substances [9].

It is still a major task in theoretical physics to understand the origin of ageing, although many different models have been proposed in recent years. Most of these are phenomenological theories like the droplet model [10], the domain growth theory [11] or the trap model [12]. A different approach is given by mean-field models of Ising spin glasses [13] and models with a hierarchical structure in phase space [14]. Most of these models are rather successful in fitting experimental data although some of them differ in their final conclusions concerning the scaling of time-dependent properties.

In order to discriminate between these differing scaling assumptions many experiments have been carried out [15], but the situation remains unclear. Therefore numerical studies play an important role, as one can obtain detailed information about the dynamical processes

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and the spatial correlations in glassy systems. Much progress has been obtained recently in the numerical studies on ageing in spin glasses and other disordered systems [16]. In numerical investigations the autocorrelation function can be calculated directly and one finds a crossover from a slow quasi-equilibrium decay for $t \ll t_w$ to a faster non-equilibrium decay for $t \gg t_w$. The functional form of these decays, e.g. for the three-dimensional Edwards–Anderson (EA) spin glass model, is algebraic and has the scaling form

$$C(t, t_w) = t^{-x(T)} \Phi_T \left(\frac{t}{t_w} \right) \tag{1}$$

with $\Phi_T(y) = c_y$, for $y = 0$, and $\Phi_T(y) \propto y^{x(T)-\lambda(T)}$, for $y \to \infty$ [16]. As an important result one gets two different exponents which are characteristic of the two dynamic regions.

Since the algebraic decay of correlation functions is also known from diffusion models in ultrametric hierarchies [17, 18], which can be solved analytically, and the analogy to the structure of Parisi's symmetry-breaking scheme in the solution of the EA-model, a hierarchical ansatz to explain ageing phenomena seems rather natural. In recent years different hierarchical models for ageing have been proposed [14]. Most of these models incorporate randomly chosen distributions of diffusion rates and, as a consequence, random energy barriers and trapping times. With further assumptions concerning the functional form of these distributions one gets ageing phenomena like the crossover from quasi- to non-equilibrium dynamics with an algebraic scaling form according to (1).

Taking long-range diffusion in an ultrametric space [17] as a starting point one can construct a hierarchical model which shows ageing phenomena as a direct consequence of the structure and not of random properties. This model will be presented in the next section followed by numerical results and some conclusions.

In contrast to the known hierarchical models for ageing, the presented model incorporates no kind of disorder or randomly distributed energies. The main idea is to take into account the different types of dynamics found in spin glasses. At short time scales $t \ll t_w$ one finds quasi-equilibrium dynamics and at long times $t \gg t_w$ non-equilibrium dynamics with a crossover region for times $t \sim t_w$. These different types of dynamics are commonly related to regions in phase space which are accessible at different time scales.

A rather natural ansatz to include these different types of states is to extend the diffusion
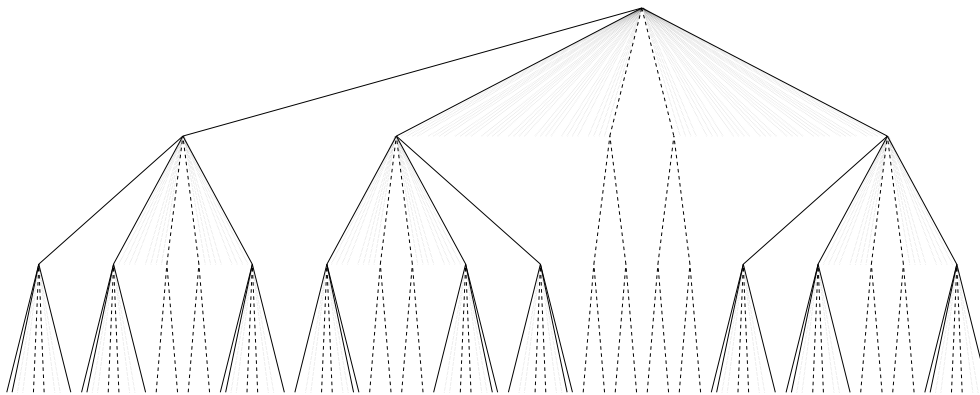

Figure 1. A hierarchical tree structure with three types of states characterized by different line styles: thick full lines represent the surrounding non-equilibrium tree, dashed lines represent the quasi-equilibrium subtrees and thin full lines represent the attractor region. The corresponding branching numbers are $n = 3, m = 2$ and $n_a = 4$; the height of the tree is $k = 3$.

on a hierarchical tree structure in which every state is equal by a more sophisticated tree structure which has at least two different types of states. The first type characterizes the quasi-equilibrium dynamics. These states are grouped as leaves in regularly branching trees of varying heights where standard long-range diffusion in an ultrametric space takes place [17]. The second class of states is found in a regular tree structure in such a way that the different equilibrium trees are embedded in this surrounding tree, i.e. every branching point of the non-equilibrium tree is the root of a quasi-equilibrium subtree. The resulting structure is shown in figure 1, where a third class of states is included in order to represent some kind of attractor states for the equilibrium trees. The overall height of the tree is k, which equals the number of different hierarchies in the tree structure. Figure 1 shows a tree of height $k = 3$ and the different trees can be recognized by the line styles, i.e. the thick full lines represent the surrounding non-equilibrium tree with the branching number denoted by $n(= 3)$, dashed lines represent the quasi-equilibrium tree with the branching number $m(= 2)$ and thin full lines represent the attractor states with the branching number $n_a(= 4)$.

The resulting diffusion rates in the master equation, defining the dynamics in the whole tree structure, are chosen to be unsymmetric which takes into account the different classes represented by the states. Let the system start at an arbitrary non-equilibrium state Q_0. We then denote the probability to be found in the whole subtree of height l ($1 \leqslant l \leqslant k$) including Q_0 with Q_l, the probability that the system can be found in a state of a quasi-equilibrium subtree which can be reached from the initial state by crossing l hierarchies with P_l and finally the probability to be found in the attractor states corresponding to the equilibrium subtree P_l with O_l. The allowed long-range hops in this tree structure are formulated in the following way [19].

(i) From a state of the non-equilibrium type every other state in the tree can be reached with a rate r_i with i being the number of hierarchies between the states.

(ii) From an equilibrium state in P_l every state within P_l can be reached with a rate r_i ($1 < i \leqslant l$) and outside of P_l every state in the tree outside of Q_l is allowed with the same diffusion rate as for the non-equilibrium states.

(iii) From the attractor states O_l hops to every state in the tree are allowed, but with the rate s_l into the corresponding equilibrium tree P_l and with the rate r_i ($l \leqslant i \leqslant k$) into all other states.

These rules include the fact that the system has to cross one hierarchy more in order to leave an equilibrium subtree because the other states within Q_l are excluded by the rules. This asymmetry is chosen to generate the equilibrium subtrees P_l as dynamical traps with characteristic escape times corresponding to their height. Using these dynamical rules the master equations can be formulated in a closed way

$$\frac{\mathrm{d}}{\mathrm{d}t} P_l(t) = -\Lambda_l P_l(t) + B_l \Omega_l(t) + B_l(Q_l(t) - P_l(t) - O_l(t))r_{l-1} + B_l s_l O_l(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} O_l(t) = -\Lambda_l O_l(t) + C_l \Omega_l(t) - B_l s_l O_l(t) - (A_l - B_l - C_l)r_{l-1}O_l(t)$$
$$+ C_l(Q_l(t) - P_l(t) - O_l(t))r_{l-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_l(t) = -\Lambda_l Q_l(t) + A_l \Omega_l(t). \tag{2}$$

To write the equations in a compact form the following abbreviations were used:

$$\Lambda_l = \sum_{i=l}^{k-1}(A_{i+1} - A_i)r_i \qquad \Omega_l(t) = \sum_{i=l}^{k-1}(Q_{i+1}(t) - Q_i(t) - P_{i+1}(t))r_i. \tag{3}$$

The constants A_l, B_l and C_l denote the number of states in a whole subtree Q_l, an equilibrium subtree P_l and the attractor states O_l, respectively. The diffusion rates r_l and s_l are as mentioned above.

The master equations can be represented by a matrix which is an upper triangular matrix despite some entries on the lower next-diagonal. In principle this system can be solved for arbitrary numbers of states and diffusion rates. In order to refer to some exact results known for the 'classical' diffusion models [17] the branching numbers of the states are taken to be constant. Therefore the number of states as defined above are

$$A_i = n^i + m \frac{n^i - m^i}{n - m} + n_a \frac{n_a^i - n^i}{n_a - n} \qquad B_i = m^i \qquad C_i = n_a^i \qquad (4)$$

with the branching numbers n, m and n_a for the non-equilibrium, equilibrium and attractor states, respectively. If the ratio of the diffusion rates is chosen to be constant, one simply gets $r_l = r^l$ and $s_l = s^l$. With this choice the master equations are solved numerically and some of the results are presented in the following section. Analytic calculations concerning the exact solution of the model can be found elsewhere [19].

Ageing phenomena are common for magnetization curves in real spin-glass materials. The corresponding function describing the main dynamical effects for spin or diffusion models is the autocorrelation function. As for magnetization experiments it is also possible to perform waiting time-dependent calculations numerically. For such purposes one has to define the correct waiting time-dependent autocorrelation function $C(t, t_w)$ which is, in the case of this hierarchical model, given by

$$C(t, t_w) = \sum_{l=1}^{k} P_{l0}(t) \sum_{i=l}^{k} c_{il} P_l(t_w, i) + Q_0(t) \sum_{i=0}^{k} c_{i0} Q_0(t_w, i) + \sum_{l=1}^{k} O_{l0}(t) \sum_{i=l}^{k} c_{il} O_l(t_w, i)$$
$$(5)$$

where $P_l(t_w, i)$ is the probability that the system has reached an equilibrium subtree of height l with i hierarchies between this subtree and the starting site after the waiting time t_w has elapsed. The constant c_{il} denotes the number of such subtrees to be found. The function $P_{l0}(t)$ describes the probability that the system is still in the same state or has returned to it after the additional time t has elapsed as it was in at the end of the waiting time t_w. The probability functions of the other states are denoted in the same way.

The waiting time-dependent autocorrelation function $C(t, t_w)$ as defined in (5) is the focal point of interest in the following. The system parameters are the branching numbers m, n, n_a and the diffusion rates r and s. These parameters can be varied in order to obtain numerical data.

Figure 2 shows the waiting time-dependent autocorrelation function $C(t, t_w)$ for parameter values as given in the caption. A crossover from slow dynamics for small times to a faster decay at large times characteristic of ageing can be seen very clearly. The crossover region is located at times $t \sim t_w$. The decay obeys an algebraic time dependence although some oscillations characteristic of self-similar systems [17] occur, especially in figure 2(a). These oscillations show a strong effect on the decay exponent which can be calculated via the logarithmic time derivative of the autocorrelation function

$$\gamma = -\frac{\partial \ln C(t, t_w)}{\partial \ln t}. \qquad (6)$$

The resulting exponent γ of an algebraic decay $C(t) \propto t^{-\gamma}$ is shown in figure 3. The strong effect of the oscillations of the autocorrelation function on the exponent can be seen in part (a) of figure 3. As shown in part (b) the mean value γ_0 of the exponent taken over one period is constant for the two different time regimes $t \ll t_w$ and $t \gg t_w$. So
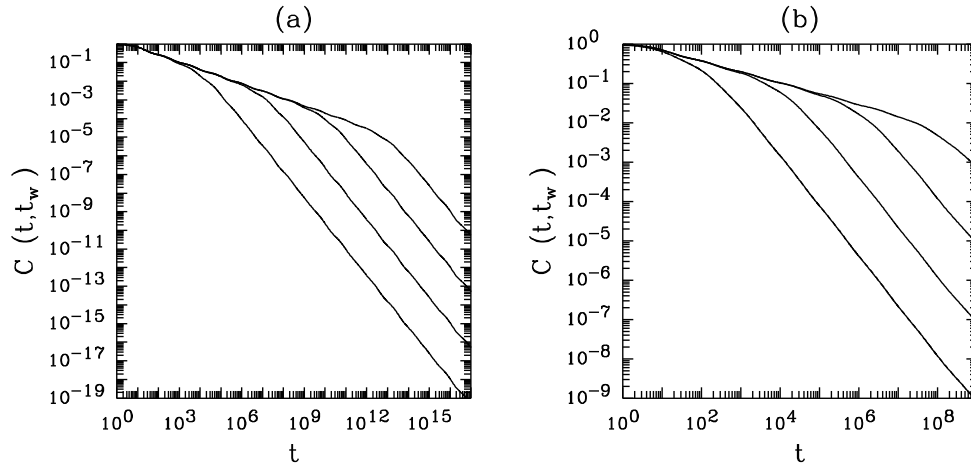
Figure 2. The waiting time-dependent autocorrelation function $C(t, t_w)$ against time t for various t_w. The system parameters are as follows: $m = 3, n = 5, n_a = 7, r = 0.02, s = 0.3$; $t_w = 10^4, 10^7, 10^{10}, 10^{13}$ in (a) and $m = 2, n = 3, n_a = 9, r = 0.04, s = 0.5$; $t_w = 10^2, 10^4, 10^6, 10^8$ in (b).
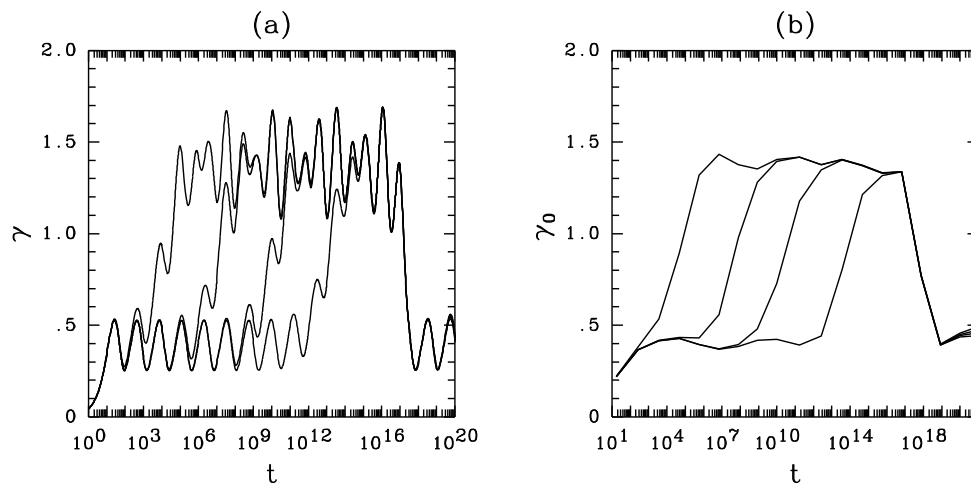

Figure 3. The exponent γ as defined in equation (6) for the autocorrelation function $C(t, t_w)$ using the same parameter values as in figure 2(a). The left diagram shows the exponent with the strong oscillations and the right diagram shows the mean value γ_0 of the exponent calculated over one period of oscillation.

the autocorrelation function satisfies the scaling relation (1) with temperature independent exponents γ_e for $t \ll t_w$ and γ_n for $t \gg t_w$.

Since the branching numbers and the diffusion rates play an important role in this model the exponents will at least depend on some of these parameters. The dynamics of this model is chosen to be close to the model solved in [17] so it is expected that the values of the exponents γ_e and γ_n will be related to the decay exponent calculated analytically for the

simple diffusion model [17], i.e.

$$\Gamma = \frac{\ln M}{|\ln R| - \ln M} \qquad \text{for } R < \frac{1}{M} \qquad (7)$$

where M is the branching number in the tree and R is the diffusion rate for crossing one hierarchy. The relation $RM < 1$ leads to an algebraic decay of the autocorrelation function, while one gets an exponential decay in the other case.
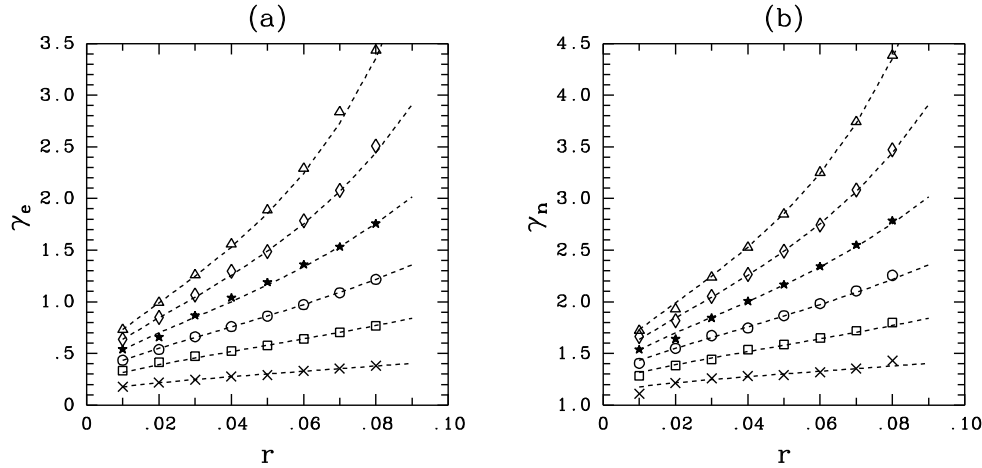

Figure 4. The exponents γ_e and γ_n against the diffusion rate r for different branching numbers m. The values of m are marked with different symbols: $m = 2$ ($\times$), 3 ($\square$), 4 ($\circ$), 5 ($\star$), 6 ($\diamond$), 7 ($\triangle$). The values of t_w and s vary, $n = 7$ and $n_a = 9$.

The exponents γ_e and γ_n for different branching numbers m and diffusion rates r are shown in figure 4. The dotted curves in diagram (a) mark the equilibrium exponent Γ according to equation (7). In the right diagram the γ-axis is shifted by one and the curves mark the values of $\Gamma + 1$. In both diagrams the numerical values are in excellent agreement with the theoretical prediction. As a consequence the values of n, n_a and s are irrelevant for the dynamical behaviour of this model. This aspect and its implications will be further discussed in the next section.

A diffusion model on a hierarchical tree structure without any disorder was introduced. It was shown that the waiting time-dependent autocorrelation function $C(t, t_w)$ shows the characteristic dependence of the waiting time t_w known as ageing in spin glasses. Since the decay of the autocorrelation function is algebraic the scaling law (1) is satisfied with two constant exponents. The influence of the parameters on the two exponents was investigated and an excellent agreement with the equilibrium exponent Γ was found for short times $t \ll t_w$. After the crossover region a non-equilibrium exponent, in good agreement with $\Gamma + 1$, was found. The dynamic interpretation can be subsumed in the following way.

• During the waiting time t_w the system gets 'trapped' in an equilibrium tree of height l corresponding to a trapping time $\tau \sim t_w$.

• Starting the measurement of $C(t, t_w)$ after the waiting time causes the system to stay in this equilibrium subtree of height l.

• The short time dynamics $t \ll t_w$ is therefore characterized by equilibrium diffusion and the equilibrium exponent Γ is found.

- In the crossover region $t \sim t_w$ the system slowly escapes out of this trap and the resulting exponent rises.
- The long time behaviour $t \gg t_w$ corresponds to non-equilibrium dynamics with an exponent $\Gamma + 1$.
- The attractor states O_l together with a large diffusion rate s forces the system into the equilibrium subtrees.
- Since the system can be found in states of the equilibrium subtrees most of the times the dynamics is completely governed by the branching number of the equilibrium states.

The presented model includes no kind of disorder but solely a hierarchical structure of phase space, so it is obvious that the hierarchical organization of traps or valleys in the energy-landscape plays an essential role in explaining the ageing phenomena. The results presented here are in good agreement with the scaling assumption (1) and coincide with the results of other hierarchical models for ageing [14] and numerical investigations [16].

Since the world of spin glasses and ageing is much richer than that discussed here, there are of course some additional phenomena which should be examined. In relation to experimental results one should consider temperature steps or more sophisticated temperature cycles. The model defined above is not directly temperature dependent, but with a straightforward relation between temperature and diffusion rates one can map temperature steps on corresponding steps in the diffusion rates [19]. This should be the focus for further investigations.

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